The Twins Paradox

The set-up minus the paradox: You have a set of twins (earth-bound Billy-Joe and space-ship Billy-Bob), both 21 years old. Space-ship Bob gets into a space ship and goes speeding off with velocity v=(24/25)c . . . (this is a relative velocity of $\beta = {}^{v}\!/_{c} = {}^{.96c}\!/_{c} = .96$). He travels for 7 years, as measured by the clock in his ship, then turns around and spends another 7 years coming home. When he arrives back at earth, how old is space-ship Bob and how old is earth-bound Joe?

To begin with, you have to decide from which frame of reference you will make your measurements. At this point, we know what is happening from the perspective of space-ship Bob's frame of reference, (we are told that his clock measures 7 years out and 7 years back, so he must be 21 + 7 +7 = 35 years old upon return), so that's the frame we will start with. Looking at earth-bound Joe from this frame, this is a standard *time dilation* problem.

Earth-bound Joe has his own set of clocks and meters sticks. He “watches” space-ship Bob (i.e., he measures Bob’s progression using his own set of meter sticks and synchronized clocks), and he registers time dilation occurring in the space ship. That is, he measures time *slowing down* in the space ship.

The amount of slow down is governed by the relationship

$$t_{earth} = \gamma \; t_{ship} \, ,$$

where t_{ship} is the elapsed time as measured by the ships clocks (14 years in this case), t_{earth} is the elapsed time as measured by the earth’s clocks (this will be related to the age of the twin who stayed behind), and γ is the relativistic factor and is equal to the relationship shown to the right:

$$\gamma = \frac{1}{\sqrt{1-\left(v/c\right)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

For this case, the relativistic factor is;

$$\gamma = \frac{1}{\left(1 - {v^2}/{c^2}\right)^{1/2}}$$

$$= \frac{1}{\left(1 - {(.96c)^2}/{c^2}\right)^{1/2}}$$

$$= 3.67$$

That means the elapsed time on earth is:

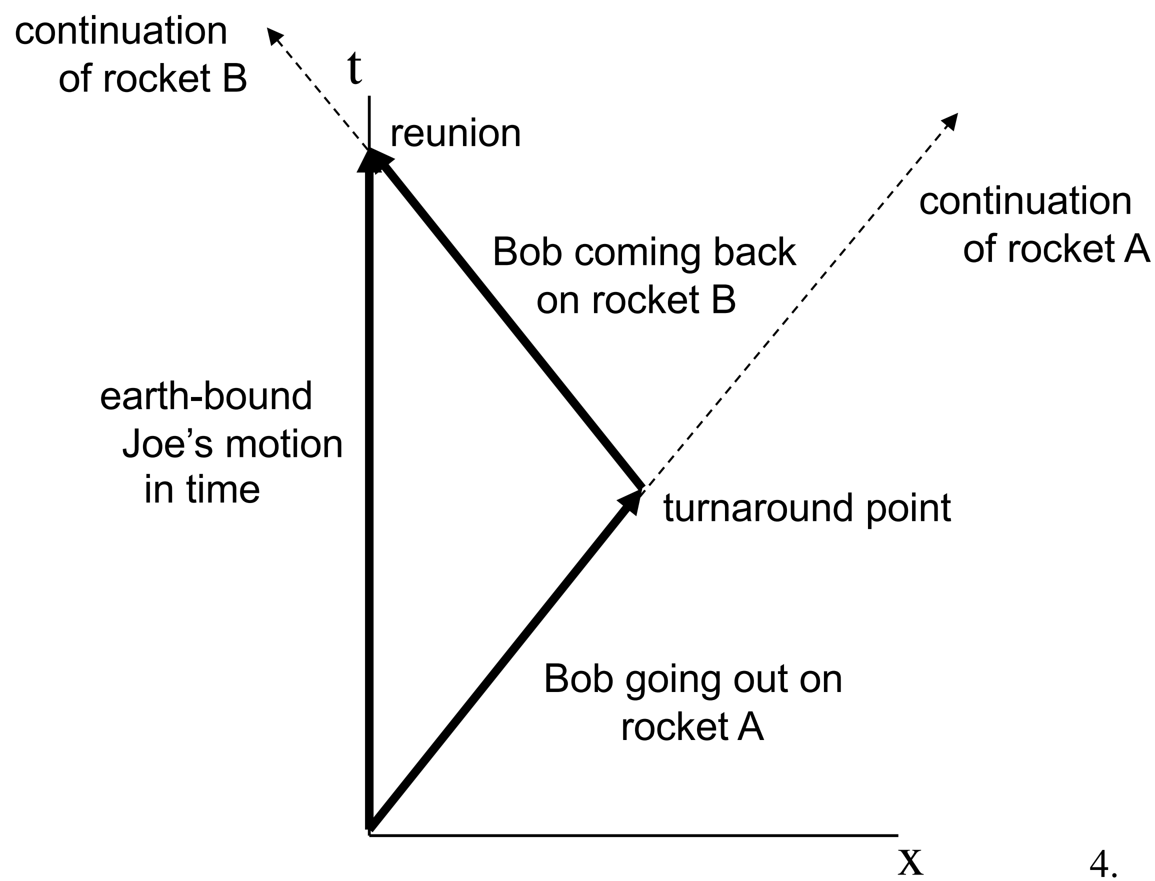
$$t_{earth} = \gamma \ t_{ship}$$

$$\Rightarrow \quad t_{earth} = (14 \text{ years})(3.67)$$

$$\Rightarrow \quad t_{earth} = 51.4 \text{ years}$$

In summary, s-s Bob was 21 years when he started the journey. He aged 14 years as measured by his ship's clock (and his own metabolic clock) during the trip. When he got back, he was 21 years + 14 years = 35 years old. E-b Joe was also 21 year old when s.s. Bob left. He aged 51.4 years as measured by his ground clock during the trip. That means that when s-s Bob got back, e-b Joe was 21 years + 51.4 years = 72.4 years old. In short, the two aged at different rates, and we still haven't gotten to the paradox.

We could look at this problem using a space-time diagram. From earth-bound Joe's frame of reference, assuming that s-s Bob took one rocket out and a return rocket back in (he has to make the turn-around somehow--we'll assume he did it this way), the diagram looks like the one to the right:



This is where things get ugly. In relativity, it is possible to transform from one frame of reference to another using what are called *the Lorentz transformations*. That means that given an x'-axis coordinate and the t'-axis coordinate for an event, you can determine the event's corresponding x-axis coordinate and the t-axis coordinate. In our case, the unprimed coordinate system is that of the earth's. YOU WILL NOT BE HELD RESPONSIBLE FOR USING THIS RELATIONSHIP ON YOUR NEXT TEST. DON'T MEMORIZE IT! For the sake of seeing the whole in all its gory details, though, I'm reproducing the *time* part of the relationship below.

$$t = x' \sinh \theta_{\text{rocket}} + t' \cosh \theta_{\text{rocket}}$$

If you would all like to join me in a rousing chorus of "YIKES," feel free. Feeling better? Good! Now to use this monster, there are a few things you need to know:

a.) The relative velocity between two objects is symbolized by a β. In this case, the relative velocity between the rocket and the earth is

$$\beta_r = {}^{v}\!/_{c} = {}^{.96c}\!/_{c} = .96$$

b.) In the kind of geometry we are working with, *cosine* functions (the side adjacent to the angle you know in a right triangle divided by the triangle's hypotenuse) gives way to what are called *hyperbolic cosines*. A similar situation occurs with the sine functions.

c.) If you mess with the geometry, it turns out that (again, nothing you need to memorize):

$$\begin{aligned}\cosh\ \theta_r &= \left(1-\beta^2\right)^{1/2}\\ &= \left(1\text{-}\left({}^{24}\!/_{25}\right)^2\right)^{1/2}\\ &= {}^{25}\!/_{7}\end{aligned}$$

where θ_r is the angle of tilt of the moving frame's axes as viewed on a space-time diagram (remember, this is related to the velocity of that frame) d.) Lastly, it needs to be noted that at the turn-around point, Bob is still at the origin of his primed coordinate axis (as far as he is concerned, neither he nor his primed axis have moved since the trip began, so his position relative to that axis has always been the same, or x'=0). Additionally, his time t' at the turn-around point, as far as the ships clock is concerned, has been given at t' = 7 years. With all that in mind, the Lorentz transformation states that the time at turn-around as measured on the earth's clocks will be:

$$
\begin{aligned}
t &= x'\sinh\theta_{rocket} + t'\cosh\theta_{rocket} \\
&= 0 + (7\text{years})\left[\left(1-\beta_r^{\,2}\right)^{1/2}\right] \\
&= (7\text{years})\left[\left(1-\left(\frac{24}{25}\right)^2\right)^{1/2}\right] \\
&= 25.7 \text{ years.}
\end{aligned}
$$

If it takes 25.7 earth years for half the trip to take place, it must take 51.4 years for the round trip to take place. In other words, the simple relativistic (time dilation) calculation we did at the start reflects the reality of the situation, at least as far as the Lorentz equations are concerned.

Isn't this fun?

Actually, it gets better 'cause we still haven't dealt with the paradox!

If any constant-velocity frame of reference is as good as any other, why doesn't Bob in the space ship look out, "see" his brother racing away from him with velocity (24/25)c, "observe" time dilation in his brother's frame and find that when he returns home, his twin, earth-bound Joe, is *younger* than he is?

That's the paradox.

Which scenario is "right" and which outcome actually happens?

It is with this bit of amusement that relativity students have been faced ever since Einstein unleashed the horror back in the early twentieth century. And it turns out that solution is conceptually simple (sort of). The reality is that s.s. Bob takes *rocket A* out away from the earth. This rocket has a set of meter sticks complete with an attached set of synchronized clocks. The whole shabang moves with Bob while he is in *rocket A*. If he does the math (the very same math we did back on page 3 but with Bob's frame being the assumed, stationary, unprimed frame), he will "observe" Joe's time to slow down just as expected for situations in which you have relative velocities. And by the time turn-around occurs, he will conclude that the passage of time on earth will have been:

$$t_{ship} = \gamma \; t_{earth}$$

$$\Rightarrow \quad t_{earth} = \frac{7 \text{ years}}{3.67}$$

$$\Rightarrow \quad t_{earth} = 1.9 \text{ years}$$

With 1.9 years passing on earth on the way out, and the same amount of time passing on the way back, it would appear that there is a discrepancy of 51.4 years minus 1.9 years minus 1.9 years, or 47.6 years. So if our first set of evaluations were correct (and remember, the Lorentz transformations did substantiate that claim), what happened to the missing 47.6 years?

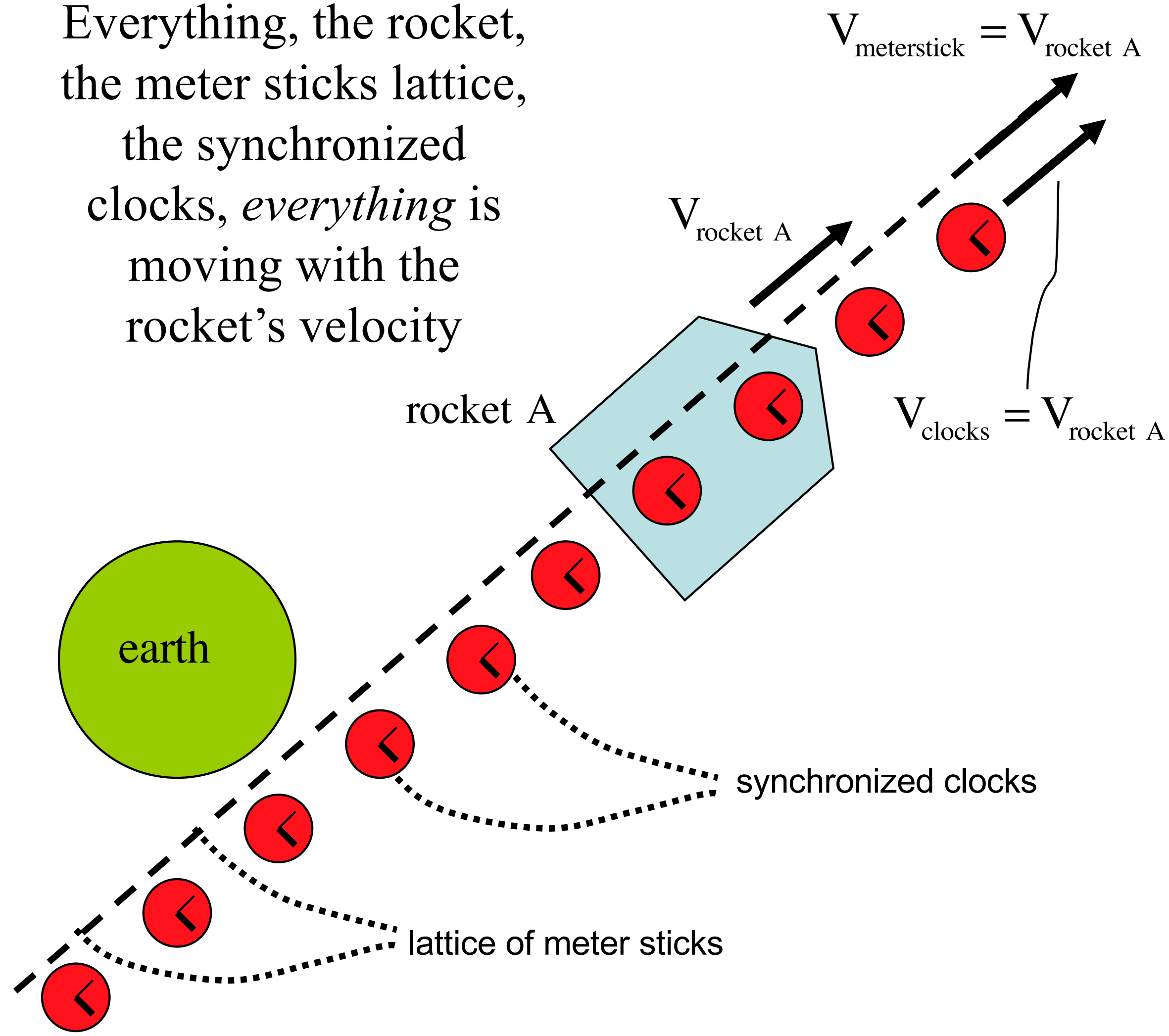
In fact, the missing years are found if we observe the consequence of space-ship Bob's transfer from *rocket A*, complete with its lattice of meter sticks and synchronized clocks, to *rocket B* with *its completely different* lattice of meter sticks and synchronized clocks. The turn-around (this is essentially an acceleration phenomenon), in other words, evidently added 47.6 years to the mix.

Interestingly, this can be seen quite nicely using a space-time diagram for the situation (and now we are back to "the fun!").

The first thing to think about is how synchronized clocks actually work. This has at its heart the question, “In relativity, what does it *really* mean to ‘observed’ an event?”

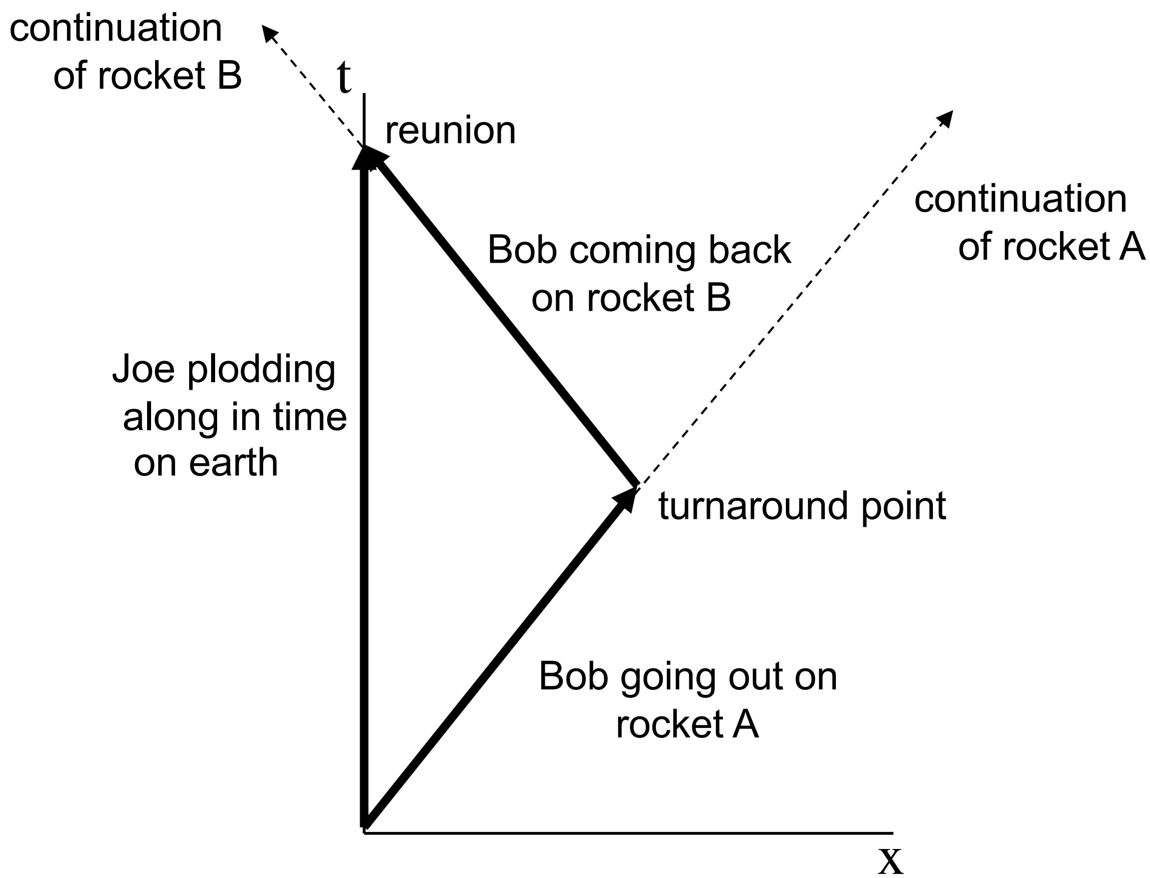
What’s actually going on with s.s. Bob’s motion? Well, he is moving along with a set of meter sticks that are rigidly attached to his outbound rocket. He isn’t moving relative to them; they aren’t moving relative to him (this is why his coordinate is always, effectively, x’=0). That lattice extends out in front of him and behind him, reaching all the way to and passed earth. Attached to each of those meter sticks is a synchronized clock.

Everything, the rocket, the meter sticks lattice, the synchronized clocks, *everything* is moving with the rocket’s velocity



nothing is to scale (obviously)

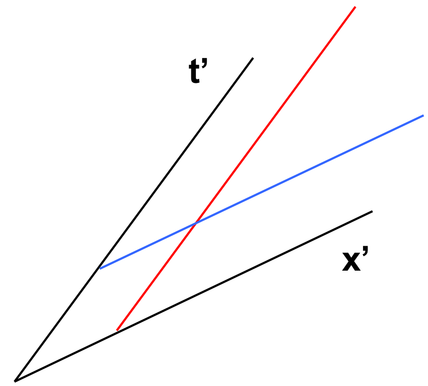
Below is the space-time diagram we generated to track s.s. Bob's world-line.



What observations can we make about the motion of both fellows?

Two observations:

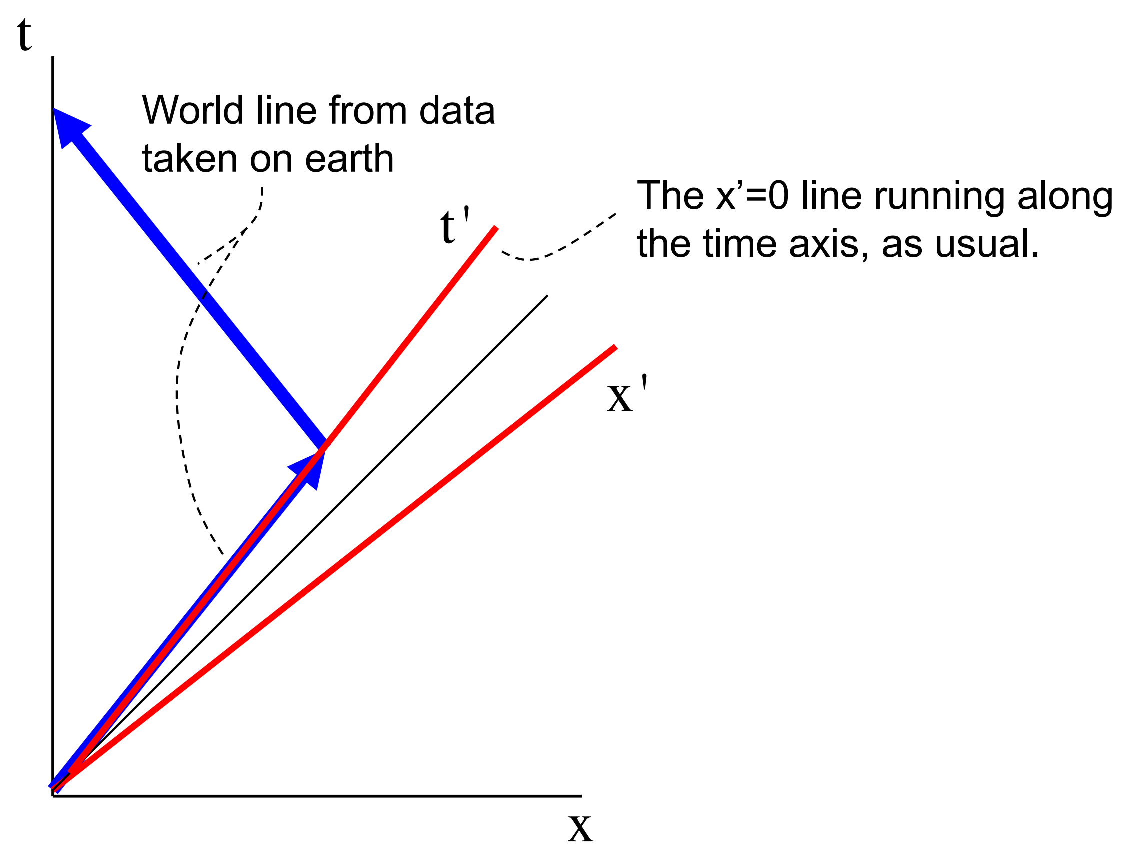
1.) Back when we were first looking at the idea of space-time diagrams, we noticed that each set of axes has lines of simultaneity that are always parallel to the x-axis of the grid.



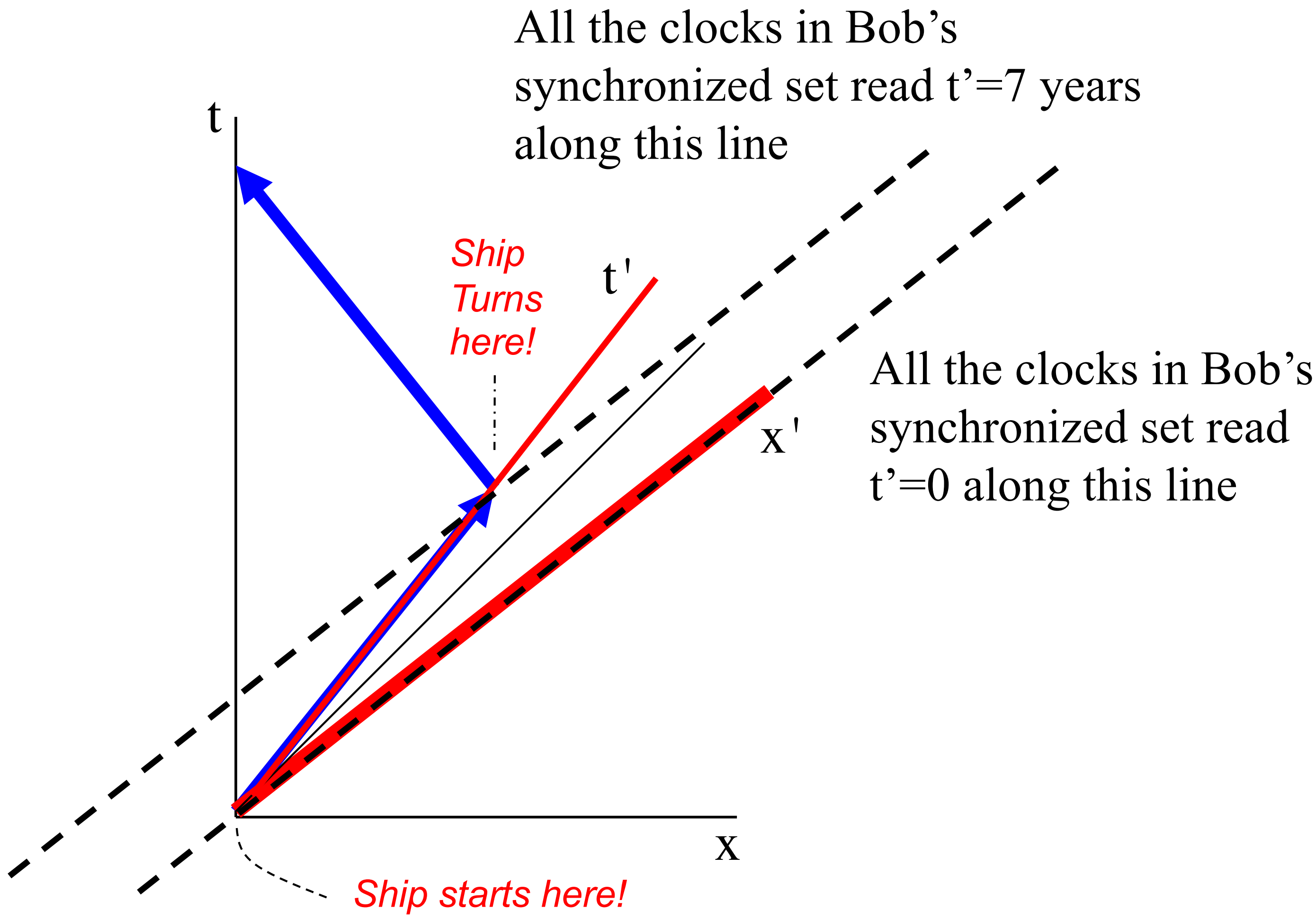
All the events that occurred at x'=3 (for example) would be found along the red line. All events that occurred at t'=4 (for example) would be found long the blue line.

2.) We actually know what s.s. Bob's x' axis looks like. We have already graphed the events associated with the space ship's motion using what we know from the earth's perspective. We also know that every one of those defined events has a primed x coordinate of x'=0 (he never leaves the origin of the primed axis as it's moving along with him). In other words, overlaying the primed coordinate axes onto our already generated world line for the space ships motion yields:

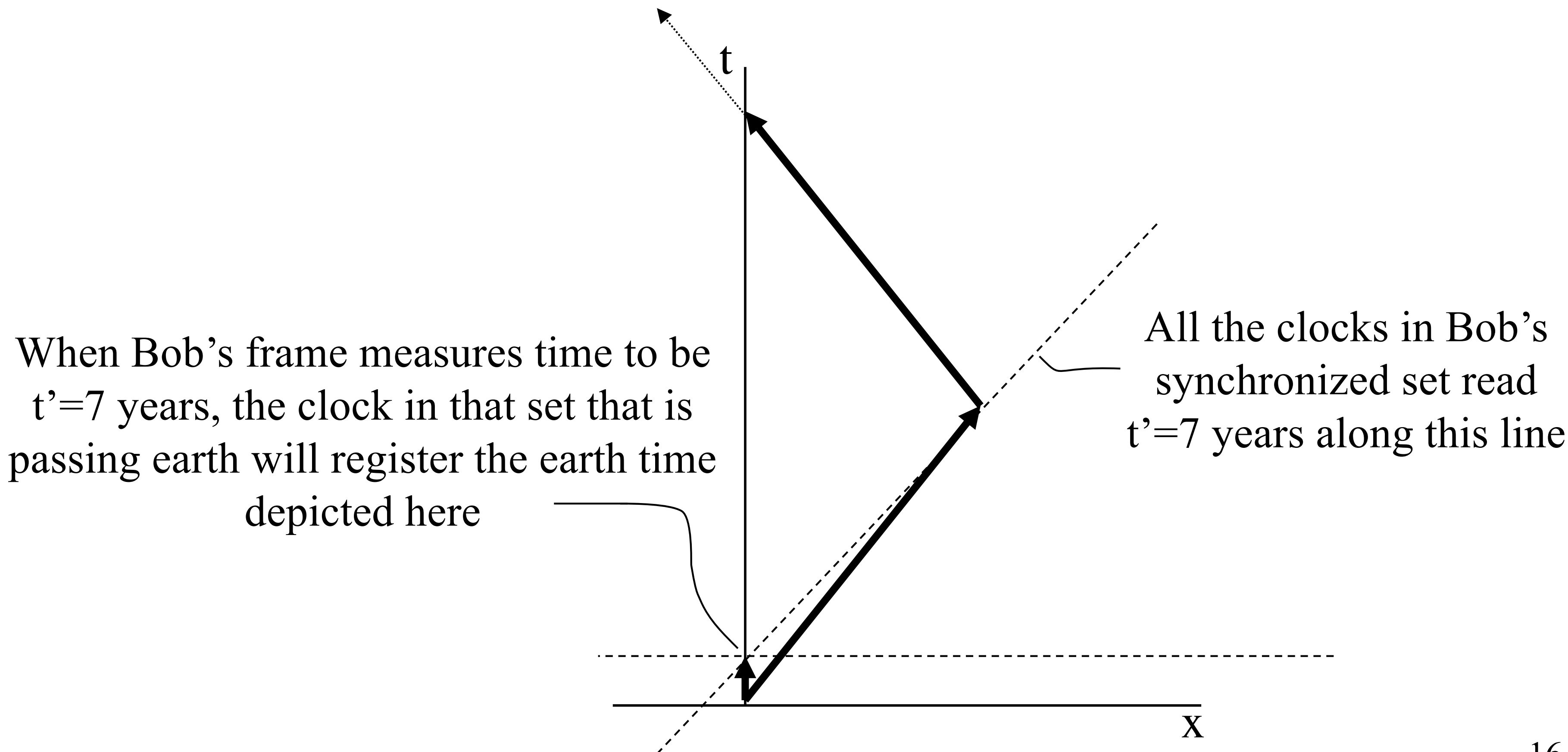
The time axis for the primed system is shown below. What’s nice is that we know there is symmetry between the positioning of the the x’ axis and the t’ axis (the angle between the time axis and the vertical is the same as the angle between the position axis and the horizontal), so we can additionally draw the x’ axis. This is all shown below.



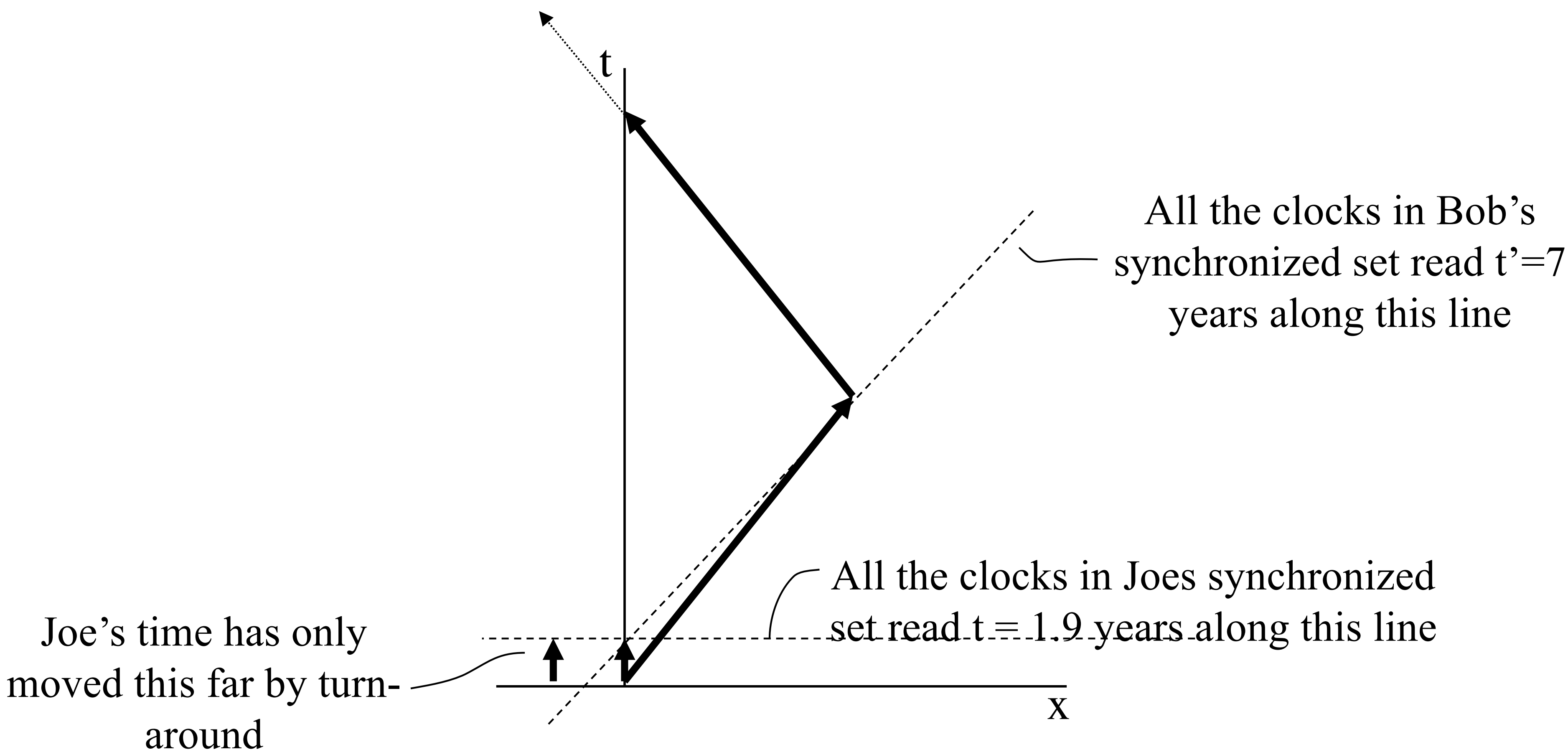
As usual, lines of simultaneity in the primed axis system will be parallel to the x' axis (the lines of simultaneity for the take-off point and for the turn-around point are shown below).



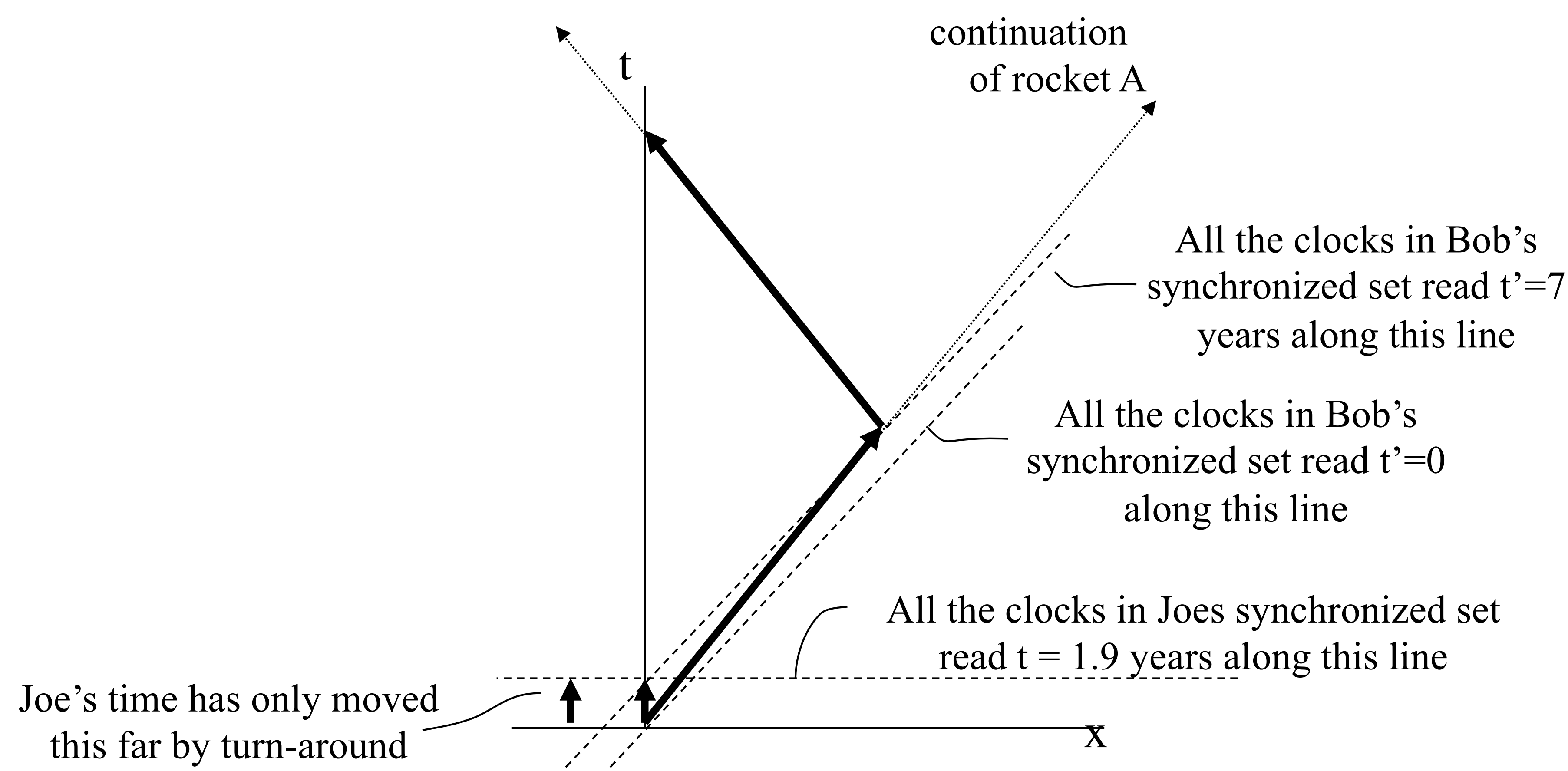
This is where you need to focus. Look at the line of simultaneity for s-s Bob's clocks at the turn-around point. There is one clock in that set of synchronized clocks that will pass the earth just as t' reaches the 7 year mark. As it does, it (being a very smart clock) will not only register its time, it will record the time showing on the synchronized set of clocks that are attached to the earth (remember, these times are measured along the vertical axis of the space-time diagram).



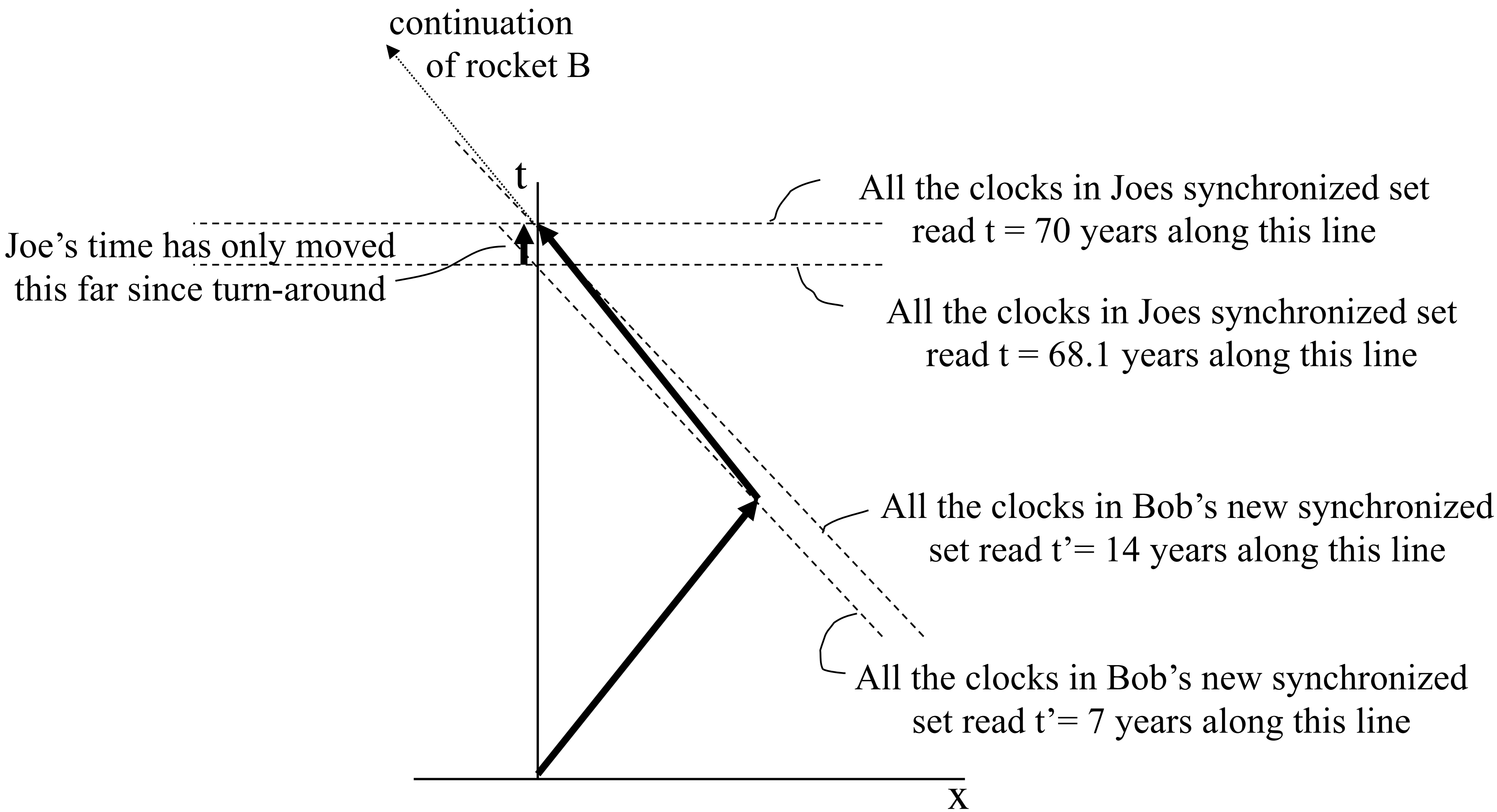
Put a little differently, on the space-time diagram, making a measurement of earth-bound Joe's time is done by moving vertically up what you and I would normally call *the y-axis*. Space-ship Bob's t'=7 years *lines of simultaneity* crosses that vertical axis somewhere. The vertical axis reading at that crossing point measures the time e-b Joe's clocks are reading at turn-around.



And to beat a dead horse completely, one more way: Just as s-s Bob's lines of simultaneity are parallel to the x'-axis, all of Joe's lines of simultaneity must be parallel to *his* x-axis. And the only line that matters to us? The one that corresponds to s-s Bob's clock reading 7 years. That line is shown below, and its time reading is only 1.9 years.



A similar analysis for the return trip yields a new set of meter sticks and synchronized clocks as shown below. Again, the elapsed time on earth between turn-around and reunion is a net 1.9 years.

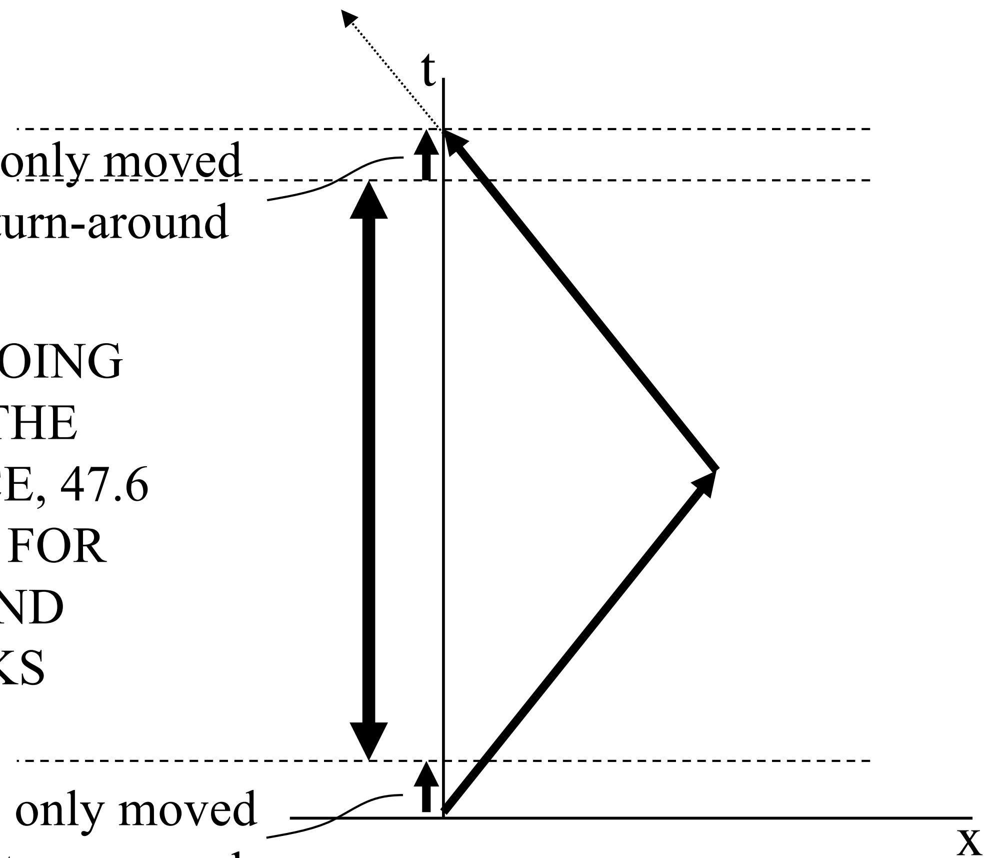


Shown below is a composite summary of the the various space-time diagrams we’ve looked at so far.

Joe’s time has only moved this far since turn-around

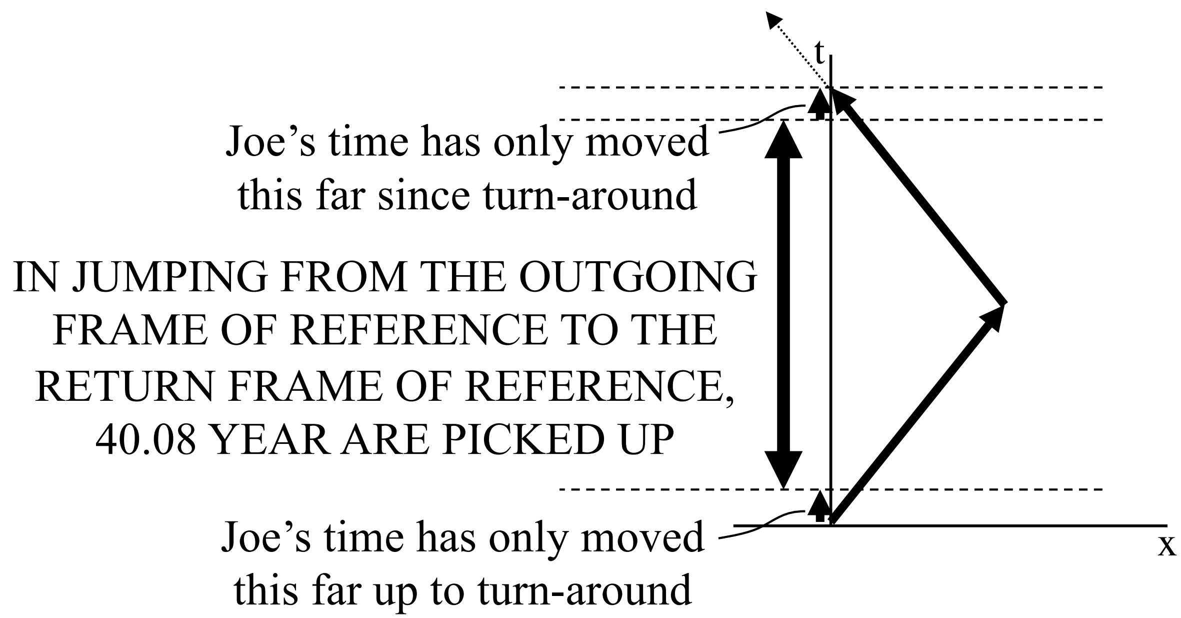
IN JUMPING FROM THE OUTGOING FRAME OF REFERENCE TO THE RETURN FRAME OF REFERENCE, 47.6 YEAR ARE NOT ACCOUNTED FOR USING BOB’S OUTGOING AND INCOMING SETS OF CLOCKS

Joe’s time has only moved this far up to turn-around



Earth-bound Joe DID have to span the time required to get to the reunion point. From our composite, it can be seen that by moving from *rocket A* to *rocket B* (and frames of reference that went with those two rockets), space-ship Bob's clocks did not track 47.6 years of elapsed time on earth.

In other words, from e-b Joe's perspective, his brother will come back as a 35 year-old while he (Joe) will be 72.4 years old. And from s-s Bob's perspective, he (Bob) will be 35 years old while his brother will be 72.4 years old. In both frames of reference, if you are consistent and don't leave anything out, you come to the same conclusion: Bob will be younger than Joe by the end of the trip.



What's more, as is often the case, simply looking at time dilation or length contraction will not solve a relativistic paradox. There will often be more going on.